Notes for: Foundations and Tools for Operations Research and the Management Sciences

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These notes are not meant to be self-contained. They are for the most part a compilation of examples -- artificial to be sure -- illustrating basic concepts and controversial points which will be discussed in the lectures. It would be ideal from my point of view, and perhaps more meaningful to the reader, if the reader would give some preliminary thought to these examples prior to the time they will be jointly discussed.

Topic 1: Decisions under risk and uncertainty.

1.1 An experiment involving objective and subjective uncertainties.

You are given two urns each containing just red and black balls. Urn No. 1 contains 50 red and 50 black balls. Urn No. 2 contains an unknown number of reds and an unknown number of blacks.

Suppose you are given two options:

Option No. 1: Select a color (R or B), announce it, and then take a single random drawing from urn No. 1.

Option No. 2: Select a color (R or B), announce it, and then take a single random drawing from urn No. 2.

Suppose further, that no matter what option you choose the payoffs are as follows:

a) If your selection differs from your drawing you get nothing.

b) If your selection agrees with your drawing you gain $100.

Answer the following questions, keeping in mind your financial position as of today.

Question 1: If you must choose either option No. 1 or option No. 2 which would you choose?

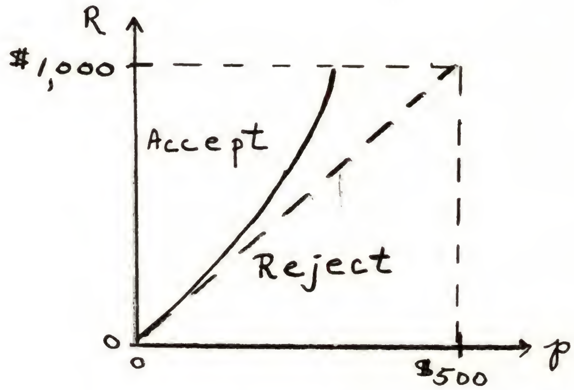
Question 2: If you were given the choice between taking no option or option No. 1 at a price, up to how much would you be willing to pay for option No. 1?

Question 3: If you were given the choice between taking no option or option No. 2 at a price, up to how much would you be willing to pay for option No. 2?

1.2. <u>An experiment involving attitudes toward money and gambling.</u>

Prepare a set of complete and unambiguous instructions for <u>your</u> agent who will act in <u>your</u> behalf. Your agent will be given just <u>one</u> option which he can either accept or reject. However, you do not know the <u>specific</u> form of the option your agent will receive except that it will be of the following <u>general</u> form: It will involve a price p for the privilege of participating in a gamble with a 50-50 chance of receiving nothing or a reward of R dollars. Such an option will be denoted by (p, R). For example, if your agent is presented with option (27, 86) and your instructions tell him to accept, then: he pays $27; a fair coin is tossed; heads he gets $86 and tails he gets nothing. You are further told R will not exceed $1,000.

Keeping your financial position, as of <u>today</u>, in mind, partition the (p, R) points into <u>your own</u> acceptance and rejection sets. For example, one possible partition might be of the form:



1.3. <u>A dynamic variation of 1.1.</u>

We start out with the identical setup of example 1.1. The new gimmick goes as follows:

Instead of a single trial there now will be a succession of ten trials. Each trial consists of the following four steps:

<u>Step 1</u>: Select a color (R or B) and announce it.

<u>Step 2</u>: Select an urn (no. 1 or 2) and draw a single random ball from the urn selected.

<u>Step 3</u>: You receive nothing if your announced color and the color of the withdrawn ball disagree; you receive $100 if they do agree.

<u>Step 4</u>: Return the withdrawn ball to its urn.

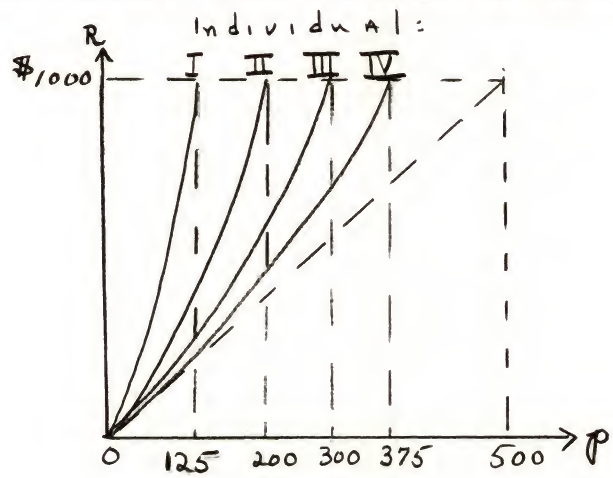
Formulate your strategy (in writing!) for the play of this ten-trial "game". Your set of instructions constitutes a bona fide strategy if and only if it satisfies this simple test: An agent acting in your behalf and using only your instructions must know exactly what to do at a given trial conditional upon his information about past trials. He should never have to return to you for further clarification of a situation not adequately dealt with in your instructions.

In this example you are being asked what you would do if you had to play. A more difficult and interesting question is: "Up to what amount would you be willing to pay for the privilege of playing this game?"

1.4. A team variation of example 1.2 illustrating risk sharing and insurance possibilities.

You are the consultant to a team of four individuals. The team has been given the task of giving a complete set of instructions to an agent, acting in the team's behalf, telling him what (p, R) options to accept and what to reject. Besides their external problem of joint acceptance or joint rejection they have an internal problem: How should they share the risks and profits accruing from an option that is jointly accepted? The internal and external problems are intrinsically intertwined and they wish you to draw up a reasonable contract for the internal behavior of the group and a set of joint instructions for their external behavior.

The individual's accept-reject strategies are given below:



The problem is a difficult one and there is no unique "answer". But nevertheless be prepared to discuss some of the guiding principles that you think might be reasonable.

1.5. An example motivating basic ideas of von Neumann Utility

Mr. Belmont must choose between two risky options. Option 1 can lead to consequences A, B, C, E with probabilities .3, .2, .1, .4 respectively; Option 2 can lead to consequences A, C, D, F with probabilities .2, .5, .2 and .1, respectively. Symbolically, we have

$$\text{Option 1: } \begin{bmatrix} A, & B, & C, & E \\ .3, & .2, & .1, & .4 \end{bmatrix},$$

$$\text{Option 2: } \begin{bmatrix} A, & C, & D, & F \\ .2, & .5, & .2, & .1 \end{bmatrix}.$$

Consequences A, B,..., F are themselves complex entities which have psychological and economic implications to Mr. Belmont.

Mr. Belmont has indicated the following personal preferences:

1) The consequences are ranked A, B, C, D, E, F from the least preferred to the most preferred.

2) Consequences B, C, D, E were each independently compared to a hypothetical option which put (1-p) weight on A and p weight on F, which symbolically is referred to

$$\begin{bmatrix} A & F \\ 1-p & p \end{bmatrix}.$$

The results of Mr. Belmont's comparisons are:

(i) B is indifferent to $\begin{bmatrix} A & F \\ .9 & .1 \end{bmatrix}$,

(ii) C is indifferent to $\begin{bmatrix} A & F \\ .6 & .4 \end{bmatrix}$,

(iii) D is indifferent to $\begin{bmatrix} A & F \\ .3 & .7 \end{bmatrix}$,

(iv) E is indifferent to $\begin{bmatrix} A & F \\ .2 & .8 \end{bmatrix}$.

Question 1: Using Mr. Belmont's preferences in 1) and 2(i) to 2(iv) which option should he choose?

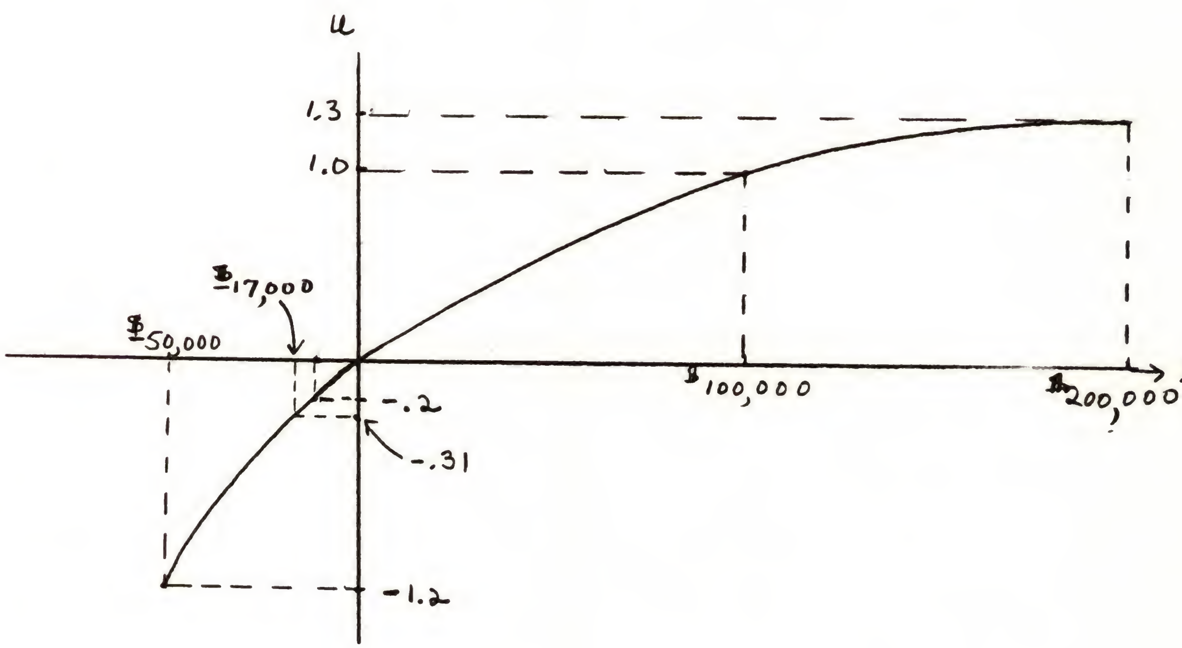
Question 2: What are the principles you invoked to reach your conclusion?

1.6. An example illustrating a utility of money curve in risk sharing.

Mr. Chelsea, is an oil wildcatter. He is constantly faced with different options whose payoffs are not certain. A typical option might be of the following variety:

Consequence:	A	B	C	D
Present monetary value of the consequence:	-\$50,000	-\$10,000	\$100,000	\$200,000
Probability	.5	.2	.2	.1

Mr. Chelsea associates to each such option a monetary, certainty, present value equivalent which is computed as follows: He first constructs a conversion function, or a utility function, which associates to each dollar amount X say, a number u(X) say, viz:



For each option he then computes the expected value of the u's and then reconverts this expected value back into a dollar certain figure. To illustrate with the above option:

Present Monetary Value of the consequence:	-\$50,000	-\$10,000	\$100,000	\$200,000	
u value	-1.2	-.2	1.0	1.3	(say)
Probability	.5	.2	.2	.1	

$$E.\ V. = (-1.2).5 + (-.2).2 + (1.0).2 + (1.3).1$$

$$= -.31.$$

Since u(-\$17,000) = -.31, Mr. Chelsea asserts he would be indifferent between giving up \$17,000 certain and taking the risky option.

Mr. Chelsea periodically reviews his utility curve -- especially when there is a change in his asset position.

Question 1: If Mr. Chelsea were offered the above option or any fraction of it he chose, and he does not have any competing capital "deals", what should he do?

Question 2: How would you go about drawing your own utility curve? To be specific suppose a single deal were offered to you with monetary consequences and associated probabilities. Not knowing the specific numbers involved (except that the payoffs are in the range from -\$1000 to +\$5000), prepare a utility function to give to your agent to act in your behalf. Remember, play the role that it is your money and you have your bills to pay at the end of the month etc.

Topic 2 Decisions leading into decisions.

2.1. Example illustrating principle of working backwards in time: A drunkard has x dollars. He spends \$1 cover charge to get into a bar, spends 1/2 of his money in the bar, and spends a dollar to get out. After five repetitions he is broke. How large is x?

2.2. A less facetious example illustrating a similar point. A warehouse problem

Mr. Medford owns a warehouse. He tries to fill his warehouse by buying commodities when costs are low and unloading his warehouse when prices are high. Assume time is discretized into periods, and each period is decomposed into a selling part followed by a buying part. Mr. Medford knows (let's divide our difficulties by assuming them away!) that the selling prices (p's) and buying costs (c's) per warehouse full of each of 3 commodities are (in thousands):

Period	1		2		3		4		5	
	p_1	c_1	p_2	c_2	p_3	c_3	p_4	c_4	p_5	c_5
Commodity 1:	-	4	9	7	3	8	6	4	7	5
2:	-	8	5	7	9	8	10	11	12	10
3:	-	11	15	13	15	12	13	11	14	19

Assume that initially the warehouse is empty and proceed on the boundary condition that it must also be empty at the end of the fifth period. Find an optimal buying and selling strategy for Mr. Medford.

Hint: Argue that:

a) There is never a need to have a product mix in the warehouse. Hence the warehouse can only be in one of four states: S_0-empty, S_1-full of commodity 1, S_2-full of commodity 2, S_3-full of commodity 3.

b) The values of being in states S_0, S_1, S_2, S_3 in the selling part of period 5 are 0, 7, 12, 14 respectively.

c) Work backward.

How does the analysis change

i) if there is a storage cost from period to period?

ii) if there is a discount factor from period to period?

iii) if Mr. Medford has a cash constraint?

2.3. A variation of 2.3 illustrating the use of natural horizons.

Mr. Arlmont has a warehouse which is equipped to handle only one commodity. He starts with an empty warehouse and knows the p_i's and c_i's for the next five periods. He does not have any constraint to have an empty warehouse sometime in the future. The data facing him is as follows.

p_1	c_1	p_2	c_2	p_3	c_3	p_4	c_4	p_5	c_5	p_6	c_6	p_7 .....
20	10	11	8	17	20	10	15	16	6	?	?	?

What should Mr. Arlmont do in the first period? Is it possible to determine what Mr. Arlmont should do in periods 2, 3, 4, and 5 without knowing p_6, c_6, p_7...?

Hint: Show that no matter what p_6, c_6, p_7,... are, it is optimal to have a full warehouse going into the selling part of the fifth period. This now works as a boundary constraint from which a backward program can start.

2.4. A decision flow chart involving chance moves.

Waco Wildcat Inc.

Honest John Waco, Wildcatter extra-ordinaire, has just finished examining a geologist's report on a corner plot of the Bumsteer Ranch. They suspect this plot to have oil in it -- someplace. Honest John has three options open to him:

1. Don't drill. Let the plot rest untapped.
2. Drill on the strength of the report he now has before him.
3. Conduct further seismological tests, then decide whether or not to drill.

In his mind he reviews the situation as follows: 'suppose I drill solely on the strength of this Rock of Ages Geological Survey Inc. report before me? Their breakdown of drilling costs and estimated present worth of oil in this plot gives me the following picture of a gamble on drilling now.

I

STATE

	Dry	Wet Natural Gas	Wet Gas and Oil	Oil
Drill now: Gain or Loss	-$100,000	$50,000	$100,000	$200,000
Subjective Probability of State	.3	.4	.2	.1

The report indicates that the estimates of 'Probability of State' are quite subjective. They are based on the geologists' weighing of four factors:

a) past experience with similar well states;

b) external examination of the Bumsteer plot;

c) number of producing wells in the near vicinity;

d) their 'hunches' as to the chances of hitting any one of the listed states.

Thus the Subjective Probabilities of State represent the geologists' willingness to place hypothetical bets as to the state of the well at odds by the bottom row if I.'

'Rather than decide on the basis of the above information alone, I can conduct further seismological tests, then make up my mind. The geologists' inform me that such tests will cost $25,000. They also inform me that such tests will not yield perfect information on the state of the plot. Here is their table picturing the venture after the tests:'

II

Conditional Probability of Seismic Information

Given the Well State

STATE

Type of Information	Dry	Wet Natural Gas	Wet Natural Gas and Oil	Oil
A	1.00	.25	0	0
B	0	.75	1.00	0
C	0	0	0	1.00
	1.00	1.00	1.00	1.00
Subjective Pr. of State	.3	.4	.2	.1

Honest John has millions in the bank. At present, this is the only deal to which he has access. What should he do?

Remarks

1. From Table II and the subjective probabilities of states, show how to derive the following table:

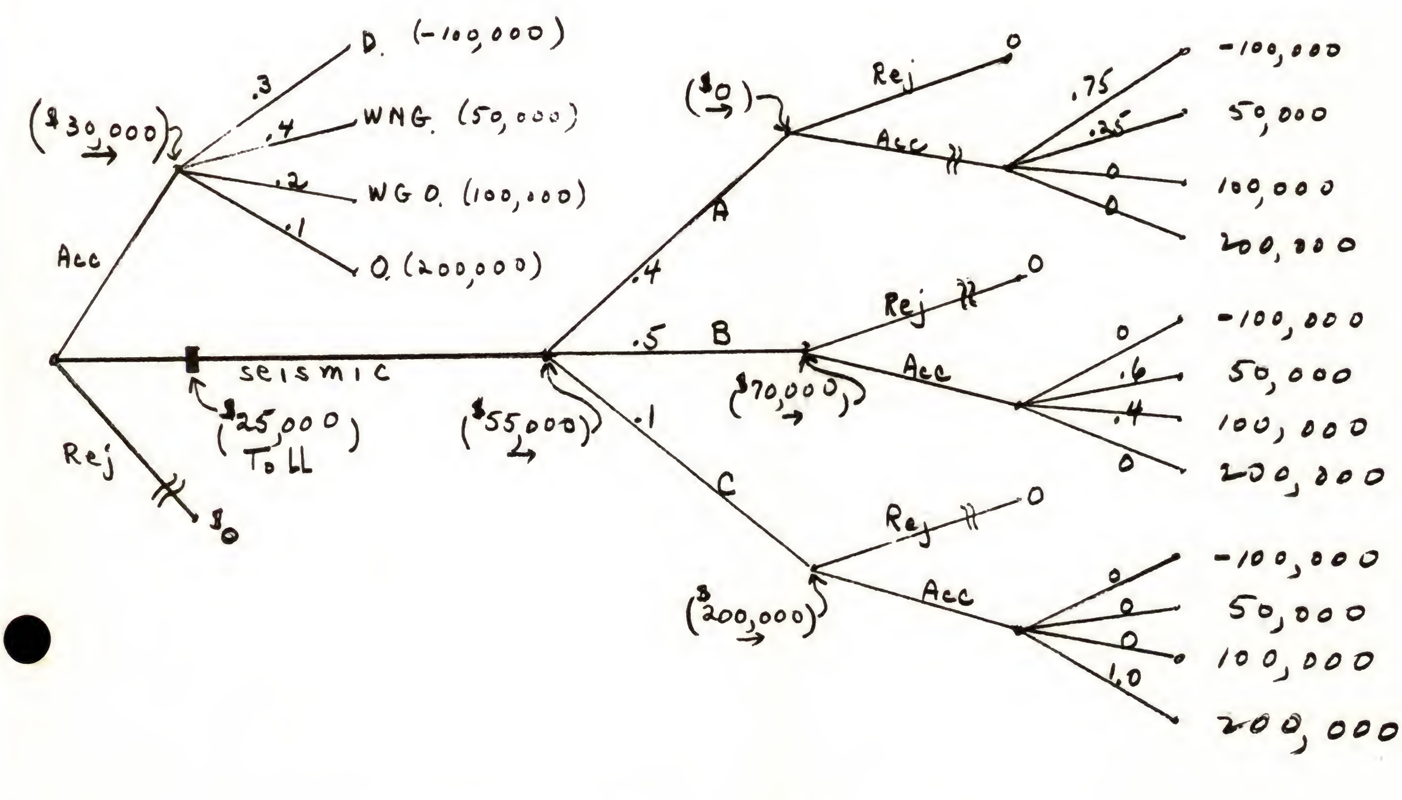
Conditional Probability of State Given Seismic Information

Type of Information

States	A	B	C
Dry	.75	0	0
Wet Nat. Gas	.25	.60	0
WNG and Oil	0	.40	0
Oil	0	0	1.00
	1.00	1.00	1.00
Probability of type of information	.4	.5	.1

(Hint: First make a table of joint probabilities of occurrences of states and types of seismic information).

2. After working 1. above, can you outline Honest John's optimal course of action for him?



Information Flow Chart

2.5. An infinite stage stationary decision problem:

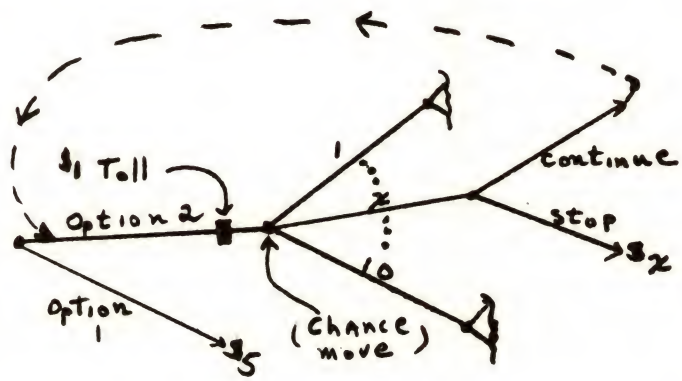
Ten balls labelled 1, 2, ...,10 are placed in an urn. Now consider the following two options:

Option 1: Draw a ball at random. If it is marked x, you receive x dollars.

Option 2: Pay \$1. Draw a ball at random. If it is marked x you have the choice of

a) receiving x dollars, or
b) returning the ball and starting over.

Which option would you choose?



Information Flow Chart

Hints:

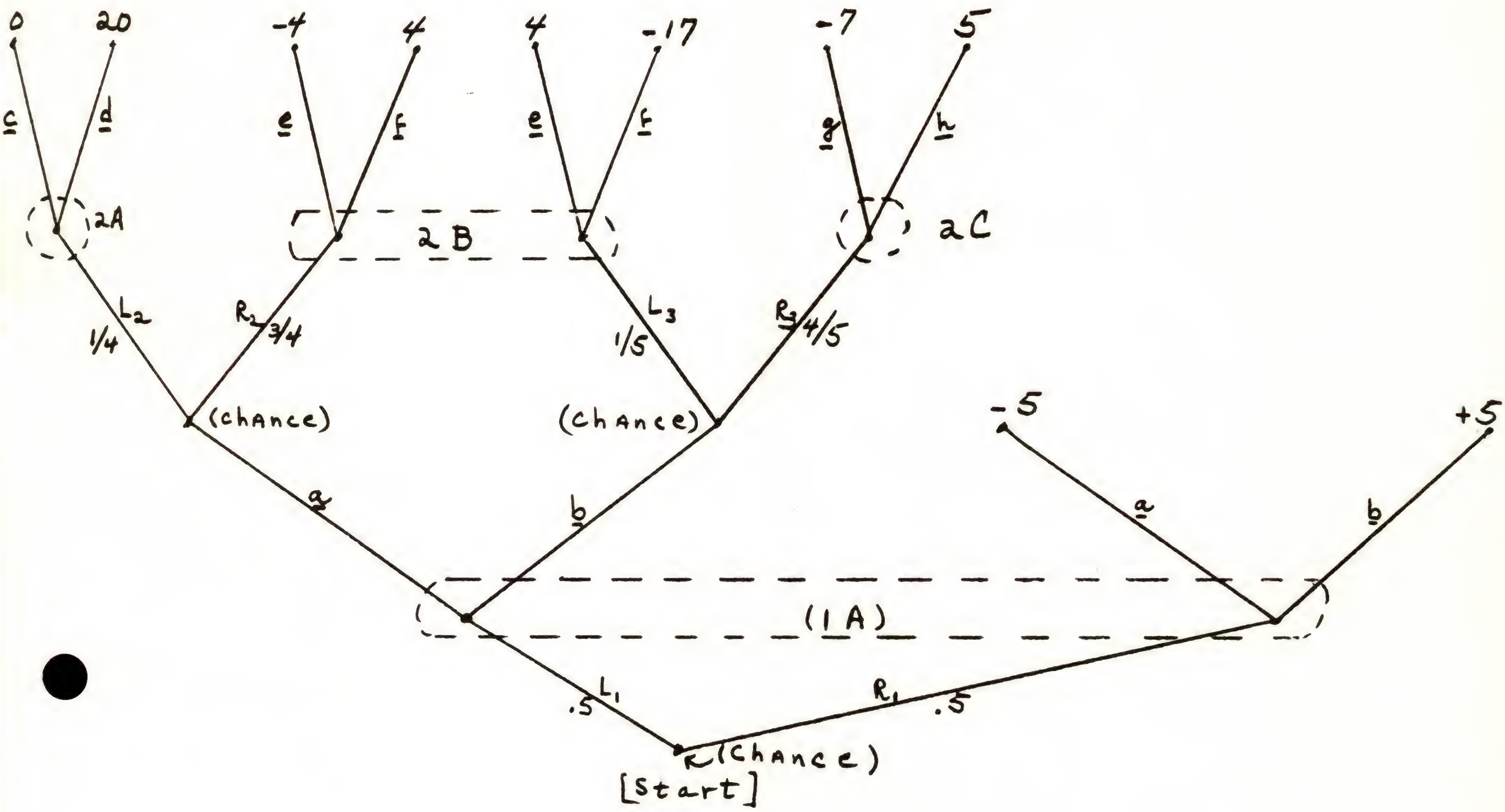
a) If v is the expected value of an optimal strategy with option 2 then in terms of v what is the optimal rule for continuation? For example if v were 5.3 would you continue sampling if x = 6?

b) The number v satisfies the functional equation:

$$v = \min_{y,\ y \text{ integral}} \left\{ -1 + v\frac{y}{10} + \frac{(y+1) + \dots + 10}{10} \right\}.$$

2.6. An (extensive) game tree.

Consider the following two-person game:



Key

Move 1: Chance move. Nature or a referee chooses L_1 or R_1 each with probability .5.

2: Pl. I's move. Pl. I is in information set 1 A. He does not know if L_1 or R_1 occurred in move 1. He has the personal choice of a or b. The game terminates after 1's move if R_1 occurs on move 1.

3. Chance Move. L_2, R_2 are chosen with probabilities 1/4, 3/4 respectively. L_3, R_3 are chosen with probabilities 1/5, 4/5 respectively.

Move 4: Pl. 2's move. Pl. 2 knows whether he is in information set 2 A, 2 B, or 2 C. If in 2 B, pl. 2 does not know whether he arrived there via (L_1, a, R_2) or (L_1, b, L_3). Pl. 2 has choices c or d in 2 A, e or f in 2 B, g or h in 2 C. Payoffs at end points represent monetary changes from Pl. 2 to Pl. 1.

Analysis of strategies:

Player 1 has one information set and two choices in it. Hence player 1 has 2 strategies which we shall label a and b. Player 2 has three information sets 2 A, B, C. A strategy for 2 consists of a choice c or d in 2 A, e or f in 2 B, g or h in 2 C. If the choices are c, e, and g say, we will denote 2's strategy by (c, e, g).

Pl. 2's strategies

		(c,e,g)	(c,f,g)	(d,e,g)	(d,f,g)	(c,e,h)	(c,f,h)	(d,e,h)	(d,f,h)
Pl. 1's	a	-4	-1	-1.5	1.5	-4	-1	-1.5	1.5
Strategies	b	.1	-2	.1	-2	4.9	2.8	4.9	2.8

Payoffs from Pl. 2 to Pl. 1

From the table of payoffs, or directly from the game tree, we note that strategies (d,e,g), (d,f,g), (c,e,h), (c,f,h), (d,e,h) and (d,f,h) are dominated.

Topic 3 Two-person games.

Reflect on the following games. In each take the role of player 1. Your adversary is a member of this class, who will remain unknown to you. You must choose a strategy from your available domain in order to attempt to maximize your utility return. But remember your adversary has similar motives.

3.1 An illustration of game-type thinking and domination:

2's strategies

1's Strategies	β_1	β_2	β_3	β_4
α_1	(2,6)	(3,3)	(12,4)	(13,2)
α_2	(4,1)	(4,1.5)	(1.0)	(4,1)
α_3	(12,0)	(3,6)	(8,-1)	(10,5)

Payoff Table

Key: For example if 1 choose α_2 and 2 chooses β_4 the payoff is (4,1) which means: 1 receives 4 'utilies of satisfaction' and 2 receives '1 utile of satisfaction.' Remember utiles are like temperature scales - they have an arbitrary origin and unit of measurement.

3.2 A zero-sum game with an equilibrium in pure strategies:

Pl. 2

Pl. 1:	β_1	β_2	β_3	β_4
α_1	18	3	0	2
α_2	0	3	8	20
α_3	5	4	5	5
α_4	16	4	2	25
α_5	9	3	0	20

Payments of Pl. 2 to Pl. 1 in Dollars

(Introduction to the Minimax Theorem).

3.3. A zero-sum game without an equilibrium in pure strategies:

Pl. 2:

$$\begin{array}{ll|ccc|}
 & & \beta_1 & \beta_2 & (3/4\ \beta_1,\ 1/4\ \beta_2) \\
\text{Pl. 1} & \alpha_1 & 3 & 1 & 5/2 \\
 & \alpha_2 & 2 & 4 & 5/2 \\
\hline
 & (\frac{1}{2}\alpha_1, \frac{1}{2}\alpha_2) & 5/2 & 5/2 & 5/2
\end{array}$$

(Statment of the Minimax Theorem)

3.4. A non-zero-sum game illustrating:

a) Efficacy of a power stragety

b) Non-interchangeability and non-equivalence of equilibrium pairs.

c) Advantage of pre-play communication.

d) Possibilities of temporal collusion.

2

$$\begin{array}{ll|cc|}
 & & \beta_1 & \beta_2 \\
1 & \alpha_1 & (2,1) & (-1,-1) \\
 & \alpha_2 & (-1,-1) & (1,2)
\end{array}$$

3.5. A non-zero-sum game illustrating efficacy of a binding pre-play threat, use of side payments and temporal policing without pre-play communication.

2

$$\begin{array}{ll|cc|}
 & & \beta_1 & \beta_2 \\
1 & \alpha_1 & (1,2) & (3,1) \\
 & \alpha_2 & (0,-200) & (2,-300)
\end{array}$$

Payments in dollars

3.6. A non-zero-sum game illustrating non-pareto Optimality of a very stable equilibrium pair. What is rationality?

$$
\begin{array}{cc|cc}
 & & \underline{\text{Pl. 2}} & \\
 & & \underline{\beta_1} & \underline{\beta_2} \\
\underline{\text{Pl. 1}} & \begin{matrix} \alpha_1 \\ \alpha_2 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} (9,9) & (-10,10) \\ (10,-10) & (-9,-9) \end{bmatrix}}
\end{array}
$$

Payments in dollars

(Play the role that your sole motivation is to maximize your own expected monetary return)

3.7. Temporal repetition of 3.6. An illustration of (a) another inadequacy of the equilibrium concept, (b) game teaching or learning.

Imagine that you are player 1 pitted against an unknown adversary (someone in this room) and that you are going to play game 3.6 not once but 5 times (trials). After each trial you will be told your opponent's choice in that trial. Since we do not have time to actually play the game 5 times, please formulate your strategy for the entire 5 trials. Your strategy must uniquely dictate your choice at any trial for any potential set of historically past choices.

Roughly how would you alter your policy if the game were to be repeated 100 times instead of 5 times.